Note VII - Contextuality Effect?

by Chenchao Ding

Nov. 18, 2024

Encoding empirical data

Support table + Bundle diagram - a minimal [possibilistic contextuality] example:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1,b_1)	1	0	0	0
(a_1,b_2)	1	0	0	0
(a_2,b_1)	1	0	0	0
(a_2,b_2)	0	1	0	0

Table 1. support table

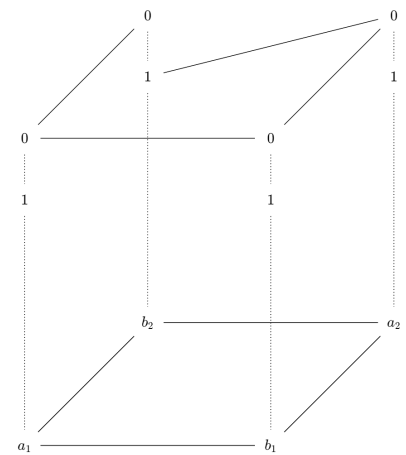
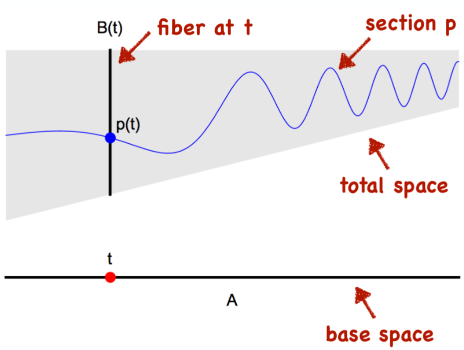


Figure 1. bundle diagram



$$\begin{aligned} X &= \{a_1, a_2, b_1, b_2\} \\ \mathcal{M} &= \{\{a_1,b_1\},\{a_1,b_2\},\{a_2,b_1\},\{a_2,b_2\}\} \\ O &= \{0,1\} \end{aligned}$$

Toward a mechanism

If we split the observable set $X = \{a_1, a_2, b_1, b_2\}$ into $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$:

$$F \ : \ \prod_{a \in A} \prod_{b \in B} O(a, b)$$

It is a "superdetermined pure function" without the contextuality effect as expected.

Instead, we want a function like this:

$$f \ : \ (x : X) \to \text{Possible}\, O(x)$$

More specifically:

- it should serve as a mechanism that produces the empirical data.
- if no specified context, the outcome for a single input x can be non-deterministic.
- if f is "simultaneously" called by x and y, the outcome for x might be determined.

Questions:

- What should this effect Possible be like?
- How to encode the context of f?

Continuation-like effect?

$$\begin{array}{rcl} \mathsf{Cont}\,R\,A & = & (A \to R) \to R \\ \texttt{call/cc} & : & ((A \to R) \to R) \to R \end{array}$$

Attempt to "capture" the co-measurement context:

$$\lambda(x\colon X).\texttt{capture}\,\lambda(g\colon (z\colon X) \to O(z)).g(x)$$

where $g\colon (z\colon X) \to O(z)$ is a partially evaluated function $g(z) = F(\_, z)$.

However, it is cheating to refer to F here, unless we admit F *is* already the mechanism. If so, it defeats our purpose to formalize contextuality effect in such a way like f.

Two main differences:

i. We expect different continuations produces different results when applied to the same value. But we expect the same result regardless of different measurement contexts!

ii. Continuation captures non-inclusive asymmetric contexts. Contextuality captures inclusive symmetric contexts.

X is a higher inductive type?

If we ignore "paths" in the base space X, "paths" in fibrations $O(x)$ are also meaningless.

$$\begin{array}{ccc} (x \in X, o \in O(x)) & \in & \coprod_{x \in X} O(x) \\ \big\downarrow & & \big\downarrow \pi \\ x & \in & X \end{array}$$

From Carlo's recent notes on dependency in category theory:

> When X is a set, maps out of X (whether element- or collection-valued) are determined simply by their values at every element of X. But in the continuous/functorial cases, the dependent type has additional data describing how each type/set/... relates to the *others* (e.g. a functorial action).

This is exactly the starting-ground to formalize contextuality. The proper view of X is (something like) a higher inductive type where there are non-trivial paths among elements of X.

```
Inductive X : Type :=
| a1, a2, b1, b2 : X
| p11 : a1 == b1 | p12 : a1 == b2 | p21 : a2 == b1 | p22 : a2 == b2
```

Neighbors and paths

$$f \;:\; (x : X) \to \mathsf{Possible}\, O(x)$$

Back to the questions:

- What should this effect Possible be like?
- How to encode the context of f?

Possible cannot just encode a collection of possible values $V \subseteq O(x)$.

Possible cannot refer back to empirical data (support table) since it is the mechanism.

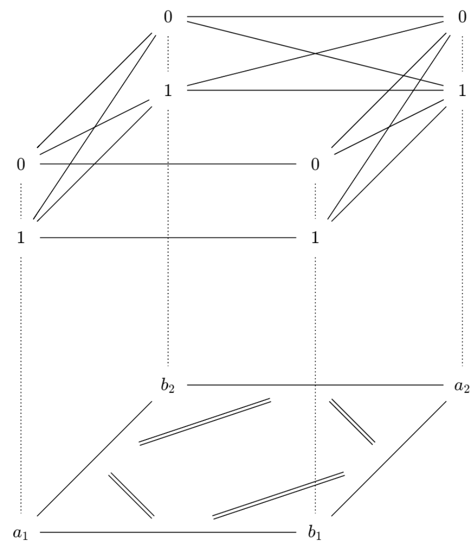
The only thing it can rely on is the topological structures (paths) in the base space X.

The concept of neighbor in the base space X is required.

$$\begin{aligned} \mathrm{tr}_O \;&:\; \prod_{x,y:X} \left(x \underset{X}{=\!=\!=} y\right) \to (O(x) \to O(y)) \\ h \;&=\; \mathrm{tr}_O(x, y, p) \end{aligned}$$

Here $h : O(x) \to O(y)$ can be a partial function or a “function with probabilistic distribution”.

Consider the [probabilistic contextuality] like this:



- paths between points in X can determine outcome $\{0, 1\}$ in $O(x)$.
- paths between paths in X can determine $\{\mathrm{id}, \mathrm{not}\}$ in $O(x) \to O(y)$.

General scenarios

We have this function with effect:

$$f \;:\; \prod_{x:X} \text{Possible}\, O(x)$$

Instead of single query input x, we can make a query regarding several variables in combination.

$$f' \;:\; \prod_{U:\mathcal{P}(X)} \prod_{x:U} \text{Possible}\, O(x)$$

If we have a function that maps a variable x to its indexed family of outcomes $O(x)$:

$$\begin{aligned} f &: (x\colon X) \to \mathcal{P}(O) \\ f &= x \mapsto O(x) \end{aligned}$$

It is equivalent to our purpose to deduce the effect on the outcomes $O(x)$ when the paths and other topological structures are imposed on X.