Note V

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Definition 1. *A measurement scenario is a triple* $\Sigma = \langle X, \mathcal{M}, O \rangle$ *where:*

- X *is a finite set of observables;*
- $\mathcal{M} \subset \mathcal{P}(X)$ *is a set of contexts, where each context* $C \in \mathcal{M}$ *represents a maximally compatible observables (or measurements that can be performed together);*
- O *is a finite set of outcomes.*

Definition 2. *A measurement cover* $\mathcal{M}$ *of a set* X *is a set of contexts s.t.:*

- *(cover)* $\bigcup_{C \in \mathcal{M}} C = X$;
- *(anti-chain) if* $C, C' \in \mathcal{M}$ *and* $C \subset C'$ *then* $C = C'$.

Example. (Bell scenario)

$$\begin{aligned} X &= \{a_1, a_2, b_1, b_2\} \\ \mathcal{M} &= \{\{a_1, b_1\}, \{a_1, b_2\}, \{a_2, b_1\}, \{a_2, b_2\}\} \\ O &= \{0, 1\} \end{aligned}$$

Event & Event Sheaf

Definition 3. *An event (assignment, section) over* $U \subset X$ *given a set of observables* X *and a set of outcomes* O *is a function* $s\colon U \to O$.

Definition 4. *An event sheaf given a set of observable* X *and a set of outcomes* O *is a functor* $\mathcal{E}\colon \mathcal{P}(X)^{\mathrm{op}} \to \mathbf{Set}$ *where:*

- $\forall U \subset X, \mathcal{E}(U) := \prod_{x \in U} O$;
- $\forall U, U' \subset X$ *and* $U \subset U'$, *a restriction map* $\mathsf{res}_U^{U'}\colon \mathcal{E}(U') \to \mathcal{E}(U)$ *of events is defined by functional restriction* $\mathsf{res}_U^{U'}(s) = s|_U$.

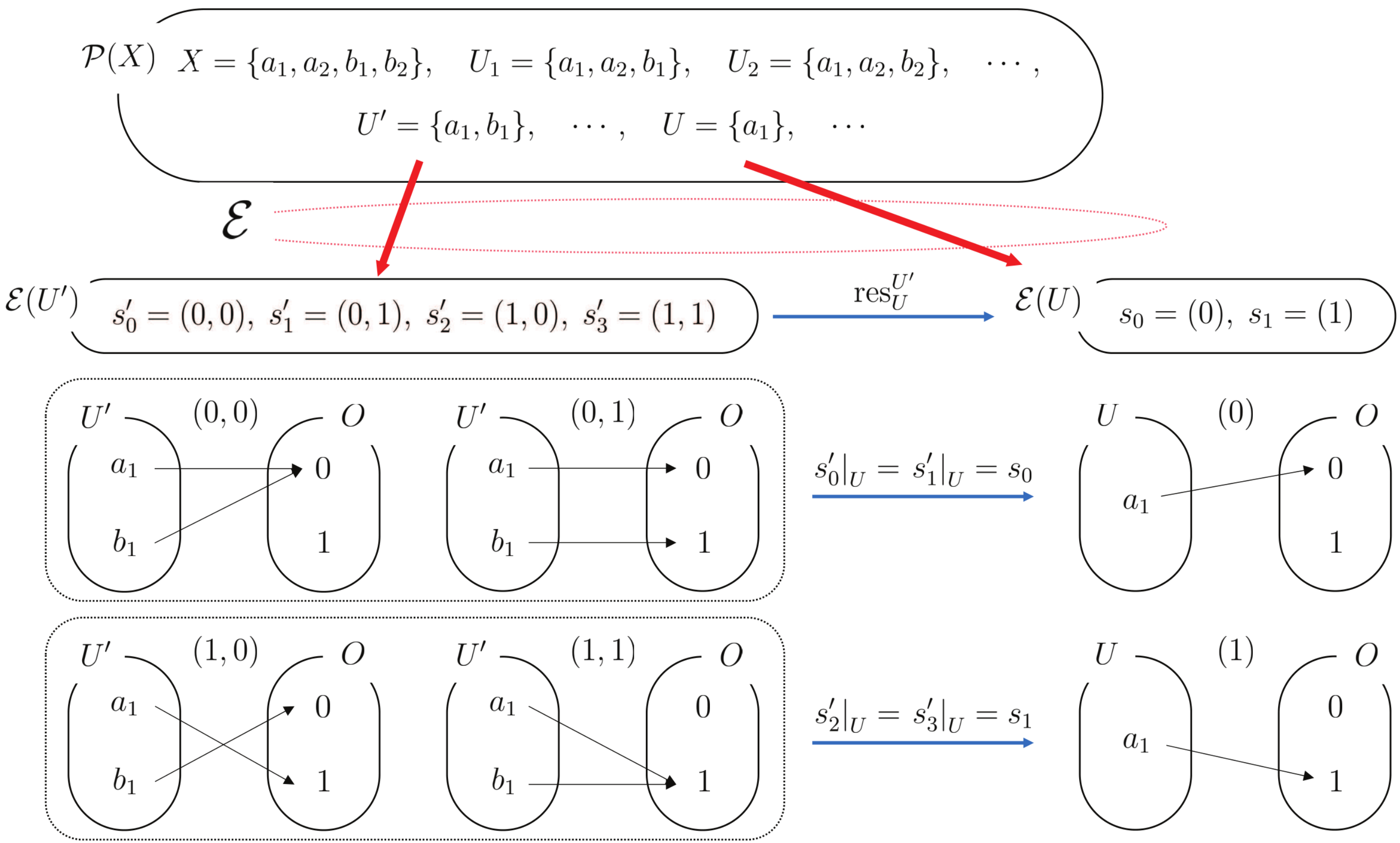
Remark. $\mathbf{Set}$ denotes the category of sets of events (maps like $s\colon U \to O$) and restricted maps.

$\mathcal{P}(X)$ denotes power set category of X whose objects are subsets of X and morphisms are inclusion maps (i.e. for $U \subset U'$, $i\colon U \hookrightarrow U'$).

$\mathcal{P}(X)^{\mathrm{op}}$ denotes the opposite category whose morphisms are projection maps instead of inclusion maps (i.e. for $U \subset U'$, $\pi\colon U' \to U$).

A projection map $\pi\colon U' \to U$ is mapped to a restriction map $\mathrm{res}_U^{U'}\colon \mathcal{E}(U') \to \mathcal{E}(U)$ by the event sheaf functor $\mathcal{E}$.

Example. (Bell scenario)



Logical Bell Inequality

Suppose N logical propositions $\varphi_1, \ldots, \varphi_N$. Each φ_i can be assigned a probability p_i.

Boolean variables appear in φ_i correspond to empirically testable quantities (observables). Each φ_i expresses a condition on the outcomes of an experiment involving these quantities. The probability p_i are obtained from the statistics of experiments.

Let $\Phi := \bigwedge_i \varphi_i$ and $P = \mathsf{Prob}(\Phi)$:

$$\begin{aligned} 1 - P &= \mathsf{Prob}(\neg\Phi) \\ &= \mathsf{Prob}\left(\bigvee_i \neg\varphi_i\right) \\ &\leqslant \sum_i \mathsf{Prob}(\neg\varphi_i) \\ &= N - \sum_i p_i \\ \sum_i p_i &\leqslant N - 1 + P \end{aligned}$$

If φ_i are carefully selected so that Φ is unsatisfiable (i.e. $P = 0$), we have:

$$\sum_i p_i \leqslant N - 1$$

Example. (Bell test)

(A,B)	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a_1,b_1)	$1/2$	0	0	$1/2$
(a_1,b_2)	$3/8$	$1/8$	$1/8$	$3/8$
(a_2,b_1)	$3/8$	$1/8$	$1/8$	$3/8$
(a_2,b_2)	$1/8$	$3/8$	$3/8$	$1/8$

Table 1. Bell test, Alice and Bob

Pick 4 conditions s.t. $\Phi := \bigwedge_i \varphi_i$ is unsatisfiable:

$$\begin{aligned}
\varphi_1 &= (a_1 \wedge b_1) \vee (\neg a_1 \wedge \neg b_1) &= a_1 \leftrightarrow b_1 \\
\varphi_2 &= (a_1 \wedge b_2) \vee (\neg a_1 \wedge \neg b_2) &= a_1 \leftrightarrow b_2 \\
\varphi_3 &= (a_2 \wedge b_1) \vee (\neg a_2 \wedge \neg b_1) &= a_2 \leftrightarrow b_1 \\
\varphi_4 &= (\neg a_2 \wedge b_2) \vee (a_2 \wedge \neg b_2) &= a_2 \oplus b_2
\end{aligned}$$

We have $p_1 = 1$ and $p_i = 6/8$ for $i = 2, 3, 4$. $\sum_i p_i = 3.25$ while $N - 1 = 3$. Violation!

Bundle Diagram

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1,b_1)	1	0	0	1
(a_1,b_2)	1	0	0	1
(a_2,b_1)	1	0	0	1
(a_2,b_2)	0	1	1	0

Table 2. PR-box support

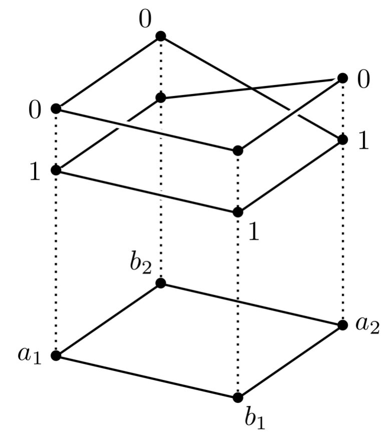
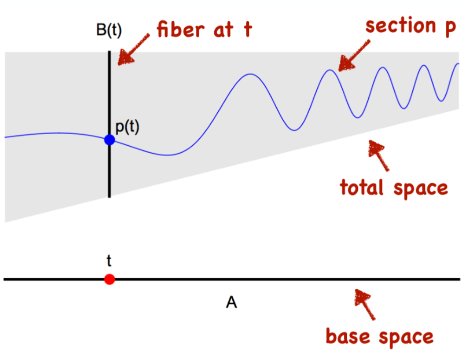


Figure 1. PR-box as bundles



(from Andrej Bauer on HoTT)

A global assignment $s_g\colon X \to O$ corresponds to a closed path traversing all the fibers *exactly once*. Such a path is called **univocal** since it assigns a unique value to each variable.

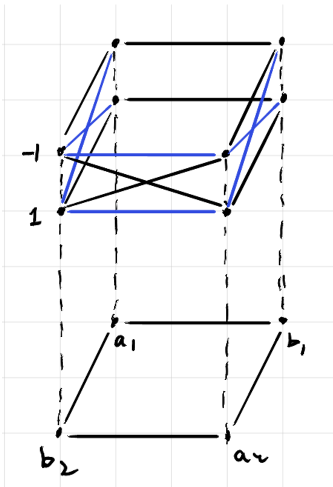


Figure 2. Bell test

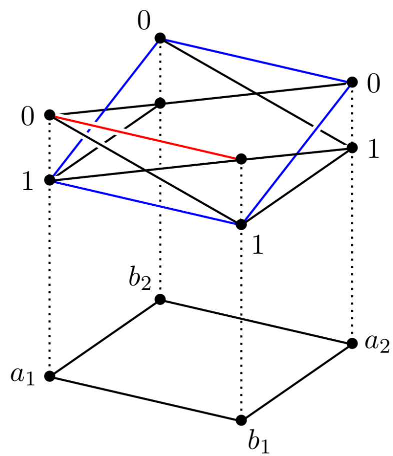


Figure 3. Hardy Paradox

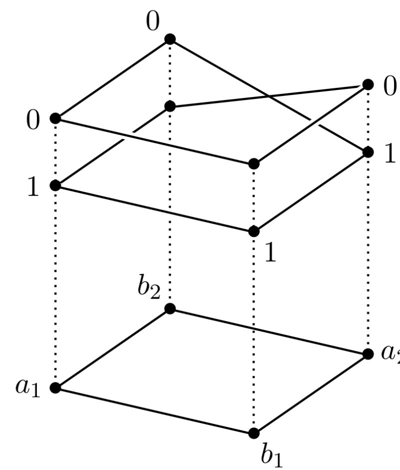


Figure 4. PR-box

$$\text{Probability (Bell)} < \text{Possibility (Hardy)} < \text{Strong (PR, GHZ, KS)}$$

- Probability: simple violation of (logical) Bell inequality.
- Possibility: at least one local section s cannot be reduced to projection of some s_g.
- Strong: there is no (consistent) global section s_g at all.

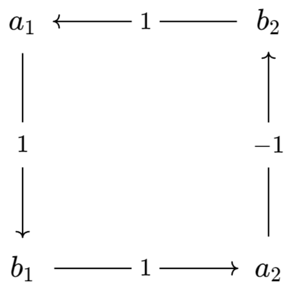
How does state-independent contextuality argument like Kochen-Specker Theorem fit into "standard model" of quantum computing (Qubit, Bloch Sphere, Pauli XYZ basis, circuits)?

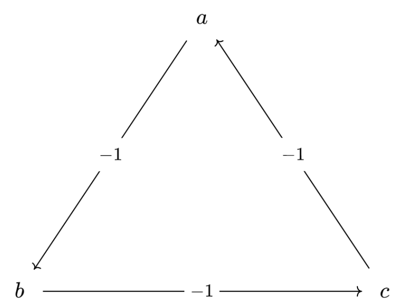
3. ALL-VERSUS-NOTHING ARGUMENTS AND PARTIAL GROUPS

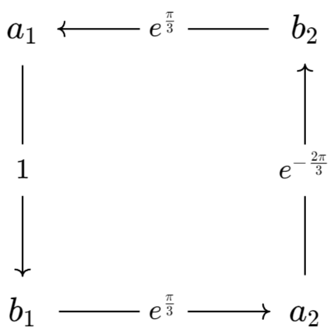
When we look into a quantum system, we see that the measurement depends on both the quantum state and the observable. However, it has been observed that some sets of observables inhere the contextuality independently from the quantum state. This type of contextuality, earlier observed by Kochen and Specker [8], has been developed to define different types of contextuality [10, 21, 23, 24]. Here, we formulate them with a sheaf-theoretic structure, starting from what is formally studied as an all-versus-nothing (AvN) argument [14, 15, 18]. We extend this argument to state-independent AvN and claim that Kochen-Specker type contextuality is, in fact, state-independent AvN in a partial closure.

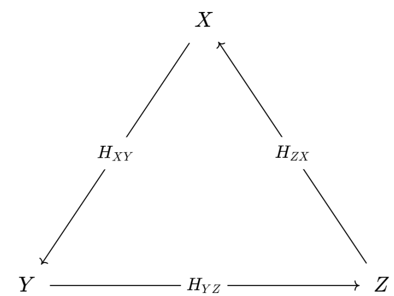
- a strange question: what is the difference between quantum state and observables?
- in "standard model" the notion of observables is hidden (in background of the whole story, Pauli XYZ), only quantum states are staged onto the "interface level".

"Base space" graphs of observables:

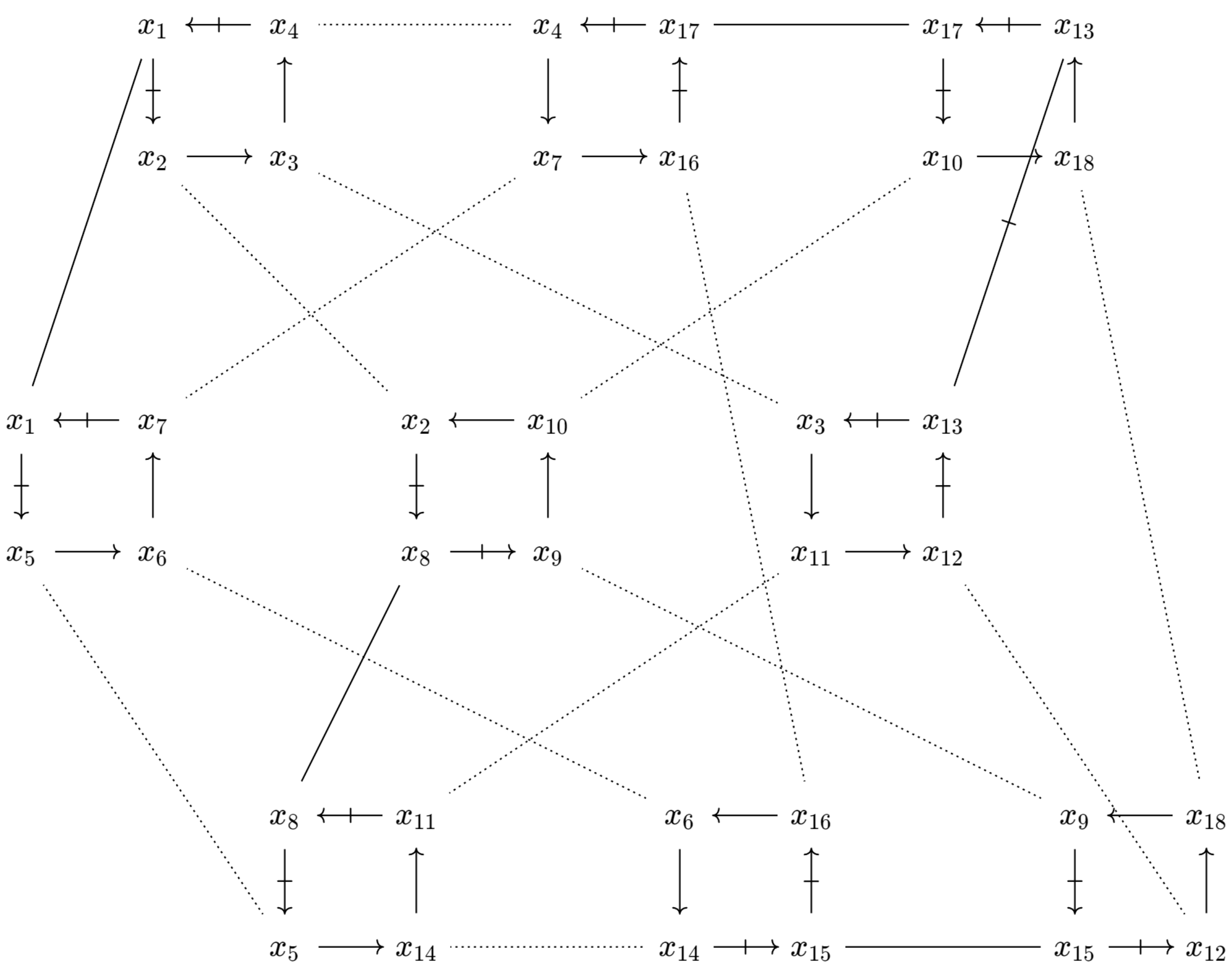








Observables can "inhere" contextuality independent of any quantum states.


x_1 x_4 x_4 x_{17} x_{17} x_{13}
x_2 x_3 x_7 x_{16} x_{10} x_{18}
x_1 x_7 x_2 x_{10} x_3 x_{13}
x_5 x_6 x_8 x_9 x_{11} x_{12}
x_8 x_{11} x_6 x_{16} x_9 x_{18}
x_5 x_{14} x_{14} x_{15} x_{15} x_{12}

Contextuality, Cohomology and Paradox

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Abstract

Contextuality is a key feature of quantum mechanics that provides an important non-classical resource for quantum information and computation. Abramsky and Brandenburger used sheaf theory to give a general treatment of contextuality in quantum theory [New Journal of Physics 13 (2011) 113036]. However, contextual phenomena are found in other fields as well, for example database theory. In this paper, we shall develop this unified view of contextuality. We provide two main contributions: first, we expose a remarkable connection between contexuality and logical paradoxes; secondly, we show that an important class of contextuality arguments has a topological origin. More specifically, we show that "All-vs-Nothing" proofs of contextuality are witnessed by cohomological obstructions.

The logic of contextuality

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Abstract

Contextuality is a key signature of quantum non-classicality, which has been shown to play a central role in enabling quantum advantage for a wide range of information-processing and computational tasks. We study the logic of contextuality from a structural point of view, in the setting of partial Boolean algebras introduced by Kochen and Specker in their seminal work. These contrast with traditional quantum logic à la Birkhoff and von Neumann in that operations such as conjunction and disjunction are partial, only being defined in the domain where they are physically meaningful.

We study how this setting relates to current work on contextuality such as the sheaf-theoretic and graph-theoretic approaches. We introduce a general free construction extending the commeasurability relation on a partial Boolean algebra, i.e. the domain of definition of the binary logical operations. This construction has a surprisingly broad range of uses. We apply it in the study of a number of issues, including:

- establishing the connection between the abstract measurement scenarios studied in the contextuality literature and the setting of partial Boolean algebras;
- formulating various contextuality properties in this setting, including probabilistic contextuality as well as the strong, state-independent notion of contextuality given by Kochen–Specker paradoxes, which are logically contradictory statements validated by partial Boolean algebras, specifically those arising from quantum mechanics;
- investigating a Logical Exclusivity Principle, and its relation to the Probabilistic Exclusivity Principle widely studied in recent work on contextuality as a step towards closing in on the set of quantum-realisable correlations;
- developing some work towards a logical presentation of the Hilbert space tensor product, using logical exclusivity to capture some of its salient quantum features.

Atom graph, partial Boolean algebra and quantum contextuality

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Abstract

Partial Boolean algebra underlies the quantum logic as an important tool for quantum contextuality. We propose the notion *atom graphs* to reveal the graph structure of partial Boolean algebra for quantum systems by proving that (i) the partial Boolean algebras for quantum systems are determined by their atom graphs; (ii) the states on atom graphs can be extended uniquely to the partial Boolean algebras, and (iii) each exclusivity graph is an induced graph of an atom graph. (i) and (ii) show that the quantum systems are uniquely determined by their atom graphs. which proves the reasonability of graphs as the models of quantum experiments. (iii) establishes a connection between partial Boolean algebra and exclusivity graphs, and introduces a method to express the exclusivity experiments more precisely. We also present a general and parametric description for Kochen-Specker theorem based on graphs, which gives a type of non-contextuality inequality for KS contextuality.

Keywords: Quantum contextuality, Partial Boolean algebra, Atom graphs, Kochen-Specker theorem

6.3 Contextual semantics

Why do such similar structures arise in such apparently different settings? The phenomenon of contextuality is pervasive. Once we start looking for it, we can find it everywhere! Examples already considered include: physics [3], computation [5], and natural language [8].

This leads to what we may call the **Contextual semantics hypothesis**: we can find common mathematical structure in all these diverse manifestations, and develop a widely applicable theory.