Contextual Observational Equivalence

Part I

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Overview

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The Varieties of Leibniz Equality

Vanilla: Leibniz Equality

Let's first take a detour, tracing back to Leibniz Equality which is the *prototypical form* of observational equivalence:

Definition (Leibniz Equality)

$$x \doteq y \stackrel{\text{def}}{=} \forall P.[\ P(x) \leftrightarrow P(y)\ ]$$

- The identity of an object x is precisely determined by a collection of properties $\{P_i\}_{i \in I}$, universally quantified by $\forall$.
- The difference between two objects x and y by default transpires in a *point*, i.e. $\neg(x \doteq y)$ IFF $\exists P.\neg[\ P(x) \leftrightarrow P(y)\ ]$.

Variant I: Set Extensionality

The first variation of Leibniz Equality is set extensionality:

Definition (Set Extensionality)

$$x = y \stackrel{\text{def}}{=} \forall z.[\, z \in x \leftrightarrow z \in y \,]$$

Replace z with a predicate $z \mapsto P_z$ where $P_z(x) = z \in x$.

- The identity of a set is precisely the sum of its elements.
- The difference of two sets necessarily transpires in a *point* (a single element), i.e. $x \neq y$ IFF $\exists z.\neg(z \in x \leftrightarrow z \in y)$

Variant II: Function Extensionality

Another variant of Leibniz Equality is function extensionality:

Definition (Function Extensionality)

$$f = g \stackrel{\text{def}}{=} \forall x. [\, f(x) = g(x) \,]$$

Replace x with a predicate $x \mapsto P_x$ where $P_x(f) = f(x)$.

- The identity of a function is precisely its point-wise graph, i.e. the sum of pairs of input/(observable) output.
- The difference of two functions f and g necessarily transpires in a *point*, i.e. $f \neq g$ IFF $\exists x. [\, f(x) \neq g(x) \,]$

Extensionality, or Well-Pointedness

In terms of category theory and specifically, topos theory:

Definition (well-pointedness)

$$\begin{tikzcd} \mathbf{1} \arrow[r, tail, "x"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Any two parallel arrows $f, g : A \to B$ are *elementally different* IFF there exists an *elementary arrow* $x : \mathbf{1} \rightarrowtail A$ telling the difference:

$$f \neq g \iff \exists x. f \circ x \neq g \circ x$$

A non-degenerate categorical universe (topos) $\mathcal{C}$ that satisfies such "arrow extensionality" is called *well-pointed.*

Ontological Decision: Atomism

So what is in common with such a constellation of similar ideas?

Ontological Atomism

An object is, by default, always viewed as the sum of its atomic elements, nothing more, and nothing less.

Consequently, there is no interesting dialectical tension or antagonism between the local (part) and the global (whole).

- It's impossible to think that two objects x and y are merely globally (or qualitatively) different, yet lacking any local (or point-wise, quantitative) evidence.
- Conversely, it's also impossible that two objects x and y are locally different, yet globally identical.

Critiques of Leibniz Equality

Critique I: Rigid Essence of Objects x amd y?

There are four layers, advancing one by one:

- **Mathematics**: objects can be freely replicated, and the function application $f(x)$ does not consume, alter, or destruct x.
- **Logic**: $A \multimap B$ encodes *ephemeral truth*, where the truth-value of A is no longer validated once B is obtained.
- **PL**: the content of a reference-object x can be modified, once assigned or passed into a function $f(x)$, witnessing *effects*.
- **Physics**: measurements involve drastic interactions between the apparatus and the objects, where the original objects can be completely destructed once measured.

Critique I: Rigid Essence of Objects x amd y?

If viewed from the strict physical perspective, the vanilla Leibniz Equality assumes objects x and y can "survive" any measurements $P(x)$ and $P(y)$ with their essence (or ipseity) intact, that is, either

- there is something like a "rigid body" within every object preserving its essential properties, with full authenticity, or
- there is a way to replicate its "rigid essence", restoring its essential properties, like money, thus minimally ideal.

In either case, it means the experimenter can *repeatedly* measure an object across multiple different configurations.

- It's the very fundamental promise of Science.
- Classically, it's done by cloning the object in concern.
- In Quantum Mechanics, however, it's not feasible to clone unknown states.

Critique II: Is Property P Pure or Effectful?

Another ontological consequence, along with the decision of atomism, is that the truth-values of $P(x)$ and $P(y)$ are immediately out there, directly accessible, as if dwelling in an atemporal bliss.

A minimal yet critical shift of perspective, following Critique I: rather than pure functions of type $\mathbf{O} \to \mathbb{B}$, the properties $\{P_i\}_{i \in I}$ should be viewed as *spatio-tmporal* effectful computations:

$$P_i : \mathbf{O} \to m\,\mathbb{B}$$

Here m is a stand-in for any concrete effect, potentially induced by, or embedded in the computational process. The properties $\{P_i\}_{i \in I}$ are now naturally like measurements or observables in physics.

Cririque III: How to Sum $\forall$ Properties?

With the presence of effects, one significant question naturally arises:

Question

Is the protocol reflecting how we *synthesize* properties quantified by $\forall$, still indifferent and neutral, specifically to *order*?

This is critical because it is the place where something more than the sum of elements comes into the picture:

Examples

Let $X = \{P_1, P_2, P_3\}$, the power set of X

$$\mathcal{P}(X) = \{\varnothing, \{P_1\}, \{P_2\}, \{P_3\}, \{P_1, P_2\}, \{P_2, P_3\}, \{P_3, P_1\}, X\}$$

encodes excessive parts more than the sum of its point-wise elements, and each part designates a unit where protocols of synthesis operate.

Cririque III: How to Sum $\forall$ Properties?

We advocate that once we shift away from ontological atomism, the protocol is never impartial, neutral, and indifferent.

Slogan

The identity of an object is captured by the sum of its elements, *plus the very act of summation (or, the very protocol of synthesis).*

A protocol can be viewed as a *configuration* of experiments, i.e. how to organize, order, group, distribute, and synthesize measurements.

- It naturally involves multi-tests, opposed to point-wise tests.
- In physics, it corresponds to the idea of compatible observables.
- In terms of set theory, we shift our focus from $\in$ to $\subseteq$.
- In topos theory, however, sub-objects are naturally primitive.

Digression: Tests or Proofs?

Theorem ($\doteq \simeq \equiv$)

The Martin-Löf identity is isomorphic to Leibniz Equality. [?]

In Martin-Löf Type Theory, the property P in Leibniz Equality is treated as a family of type $A \to \mathcal{U}$ indexed by A.

$$
\begin{aligned}
\_ \doteq \_ &: (a\ b : A) \to \mathcal{U}_1 \\
x \doteq y &= \forall (P : A \to \mathcal{U}) \to (P\ x \to P\ y)
\end{aligned}
$$

The proposition demands a potential proof of it, instead of a test, or a measurement. It does not result in a bivalent truth-value of type $\mathbb{B}$, but any type-value that dwells in the universe $\mathcal{U}$.

Contexts and Measurement Scenarios

Formalization I: From Set to Topological Space

Definition (topology, topological space)

Let X be a set. $\mathcal{P}(X) = \{U \subseteq X\}$ denotes its parts. A *topology* on X is $\mathbf{Op} \subseteq \mathcal{P}(X)$, whose elements are called *open sets*, satisfying:

- **Whole set**: $X \subseteq X$ is open, i.e. $X \in \mathbf{Op}$.
- **Binary intersections**: If $U, V \in \mathbf{Op}$, then $(U \cap V) \in \mathbf{Op}$.
- **Arbitrary unions**: If $\forall (i \in I).U_i \in \mathbf{Op}$, then $\bigcup_{i \in I} U_i \in \mathbf{Op}$; specifically, when $I = \varnothing$, we have $\varnothing \in \mathbf{Op}$.

A pair $(X, \mathbf{Op})$ is called a topological space.

In addition to the story of "local vs. global", the topological perspective bestows upon us with "localization vs. globalization".

Formalization I: From Set to Topological Space

We start with the formal definition of *measurement scenario* presented by Abramsky and Brandenburger [?]

Definition (Measurement Scenario)

A measurement scenario is a triple $\Sigma = \langle X, \mathcal{M}, \Omega \rangle$, where

- X is a finite set of observables (properties).
- $\mathcal{M} \subset \mathcal{P}(X)$ is a set of contexts that *covers* X, that is, $\cup_{C \in \mathcal{M}} C = X$. Each $C \in \mathcal{M}$ designates a measurement context, within which properties can be measured together (commuting, compatible, co-measurable).
- Ω stands for a finite set of observable outcomes or signals.

Formalization II: From Function to Functor

Definition (Section)

A section (or *event*) over $U \subseteq X$ given a set of finite observables X and a set of outcomes Ω, is a function $s : U \to \Omega$.

Definition (Global Section)

A global section is a section when $U = X$, that is, a function $s : X \to \Omega$ that assigns each observable in X a definite value.

Definition (Event Sheaf)

An event sheaf given a set of observables X and a set of outcomes Ω, is a *functor* $\mathcal{E} : \mathcal{P}(X)^{\mathrm{op}} \to \mathbf{Set}$, where

- For all $U \subset X$, there is $\mathcal{E}(U) := \Pi_{x \in U} \Omega = \Omega^U$.
- For all $U, V \subset X$ and $U \subseteq V$, there is a restriction map $\mathrm{res}|_U^V : \mathcal{E}(U \hookrightarrow V) := \mathcal{E}(V) \to \mathcal{E}(U)$.

Formalization II: From Function to Functor

The commutative diagram for $\mathcal{E}$ is shown below, where π_1 stands for a projection map from V to U when $U \subseteq V$.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
U & V \arrow[l, "\pi_1"'] & W \arrow[l, "\pi_2"'] \arrow[ll, bend right, "\pi_{21}"'] \\
\mathcal{E}(U) \arrow[from=u, "\mathcal{E}" description] & \mathcal{E}(V) \arrow[from=u, "\mathcal{E}" description] \arrow[l, "\mathrm{res}|_U^V"'] & \mathcal{E}(W) \arrow[from=u, "\mathcal{E}" description] \arrow[l, "\mathrm{res}|_V^W"'] \arrow[ll, bend left, "\mathrm{res}|_U^W"]
\end{tikzcd}
$$

As we will see later, the functor $\mathcal{E}$ maps each region $U \subseteq X$ to a (distribution of) "regional evaluator(s)", that is, $U \mapsto (U \to \Omega)$.

Refined Equality I: Introducing Contexts

Definition (*Contextual Leibniz Equality)

Let $\Sigma = \langle X, \mathcal{M}, \Omega \rangle$ be a measurement scenario:

$$(x \doteq y)_\Sigma \stackrel{\text{def}}{=} \forall (C \in \mathcal{M}).[\, \mathsf{ev}_C(x) = \mathsf{ev}_C(y) \,]$$

where $\mathsf{ev}_C(x), \mathsf{ev}_C(y) : (C \to \Omega) \simeq \Omega^C$ denotes a *joint evaluation* of all compatible observables within the context C.

- From the perspective of proofs rather than tests, the equality of two joint evaluations, namely $\mathsf{ev}_C(x)$ and $\mathsf{ev}_C(y)$, since they are functions, still requires function extensionality.
- This strongly implies that a context C is like a snapshot, where things are locally point-wise, or locally classical.

Refined Equality I: Introducing Contexts

Therefore, we can define a "localized" version of Leibniz Equality:

Definition (Localized Leibniz Equality)

Let C be a measurement context:

$$x \doteq_C y \stackrel{\text{def}}{=} \forall(P \in C).[\, \mathsf{ev}_C(P)(x) = \mathsf{ev}_C(P)(y) \,]$$

where $\mathsf{ev}_C : (P \in C) \to (x : \mathbf{O}) \to \Omega$. It can be parameterized over C, if we restrict the context C within a specific scenario Σ:

$$\mathsf{ev} : (C \in \mathcal{M}) \to (P \in C) \to (x : \mathbf{O}) \to \Omega$$

Consequently, we can use it to re-express CLE:

Definition (*Contextual Leibniz Equality)

$$(x \doteq y)_\Sigma \stackrel{\text{def}}{=} \forall(C \in \mathcal{M}).[\, x \doteq_C y \,]$$

Refined Equality II: Arbitrary Regions U, V

Definition (*Contextual Leibniz Equality)

Let $\Sigma = \langle X, \mathcal{M}, \Omega \rangle$ be a measurement scenario, and $U \in \mathcal{P}(X)$:

$$(x \doteq_U y)_\Sigma \stackrel{\text{def}}{=} \forall (C \in \mathcal{M})(C \subseteq U).[\, x \doteq_C y \,]$$

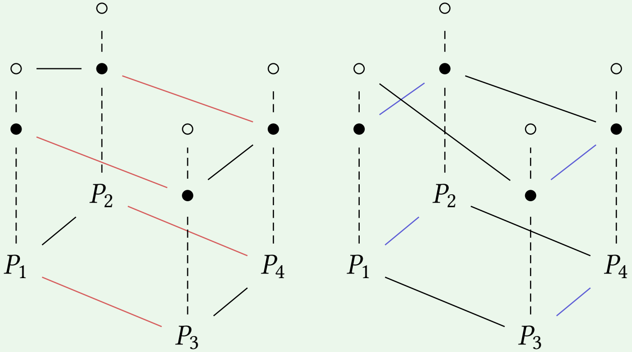
Specifically,

- when $U = \varnothing$, we obtain $(x \doteq_\varnothing y)_\Sigma$ as vacuously true;
- when $U = X$, we obtain the global test $(x \doteq y)_\Sigma$.

- Each region U refers to a distribution of testing tasks.
- It's possible to merge two regions $(x \doteq_{U \cup V} y)_\Sigma$.
- It's possible to merge two scenarios $(x \doteq_U y)_{\Sigma_1 \dot{+} \Sigma_2}$.
- In the end, it's possible to define a "topology of evaluators".

Example: Failure of Point-wise Distinction

Examples



The vanilla Leibniz Equality can be feigned by choosing different covers that do not overlap. It is forbidden in CLE.

Analogously, we can say it's not "well-pointed", since the distinction between two objects cannot be told by any elementary point.

Considering Truth-Values of $(x \doteq_U y)_\Sigma$

Question

If $(x \doteq_U y)_\Sigma$ is viewed as a computation process that simulates some quasi-physical experiments, rather than a proposition that demands proofs, then what is the potential outcomes? Or specifically, should it even be a total function?

A simple thought on merging two regions with an shared point P:

- If they disagree on P, it means *undefined*, i.e. $\perp$.
- If they agree on P, it requires further tests and validations.

The current definition of CLE still misses something:

- It treats contexts $C \in \mathcal{M}$ differently than arbitrary regions U, V.
- So far it has not yet placed the effect m into the picture. Maybe partiality $\perp$ is a most straightforward way to integrate m.

Contextuality via Effect?

The Ambiguity of m

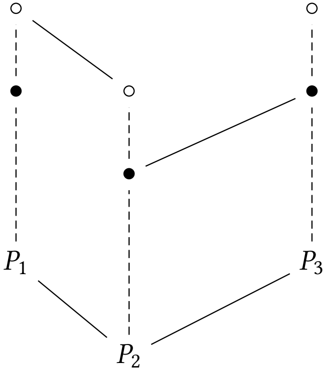
Question

Whether (or how much) the effect should be observable?

- **Observable**: m accounts for building the barrier of compatibility, or “co-measurability”, among observables.
- **Unobservable** (nonlocal hidden variables): m accounts for the contextuality and nonlocality that show up in data.

Example: Maybe Monad as Partiality

Consider the following bundle diagram:



Question

What's the evaluation at point P_2?

Example: Maybe Monad as Partiality

$$
\begin{tikzcd}
 & \{P_1, P_2, P_3\} & \\
C_1 = \{P_1, P_2\} \arrow[ur, hook, "i_{12}"] & \{P_2\} \arrow[u, hook, "p"] \arrow[l, hook', "p_l"'] \arrow[r, hook, "p_r"'] & \{P_2, P_3\} = C_2 \arrow[ul, hook', "i_{23}"']
\end{tikzcd}
$$

$$
\begin{aligned}
\mathsf{ev}_{C_1} &: (P \in C_1) \to (x \in \mathbf{O}) \to \Omega \\
\mathsf{ev}_{C_2} &: (P \in C_2) \to (x \in \mathbf{O}) \to \Omega \\
\mathsf{ev}_{C_1 \cup C_2} &: (P \in C_1 \cup C_2) \to (x \in \mathbf{O}) \to \mathsf{Maybe}\ \Omega
\end{aligned}
$$

Here a nonlocal section are viewed as a *partial* function. The effect can emerge when you try to “join” two local sections indexed by context.

Restriction and Partial Evaluation

Vertical and Lateral Composition of Evaluators

Reformulation in Topos Theory

Centration of Sub-objects of O

$$
\begin{tikzcd}
\mathbf{1} \arrow[dr, tail, "a"] \arrow[drr, tail, "o"] \arrow[ddr, tail, "id_{\mathbf{1}}"'] & & \\
& A \arrow[r, tail, "f"'] \arrow[d, dashed, tail, "!"] & \mathbf{O} \arrow[d, "P_A"] \\
& \mathbf{1} \arrow[r, tail, "\mathsf{t}"] & \Omega
\end{tikzcd}
$$

In this commutative diagram:

- The object $\mathbf{O}$ corresponds to the collection of all objects.
- The arrow $f : A \rightarrowtail \mathbf{O}$ is a sub-object (part) of $\mathbf{O}$.
- The centration (characterization) of f is a *property* $P_A : \mathbf{O} \to \Omega$.
- The arrow $o : \mathbf{1} \rightarrowtail \mathbf{O} \in f : A \rightarrowtail \mathbf{O}$, witnessed by the monic $a : \mathbf{1} \rightarrowtail A$. The equation $P_A \circ o = \mathsf{t}$ says o satisfies P_A.

Centration of Sub-objects of $\Omega^{\mathbf{O}}$

$$
\begin{tikzcd}
\mathbf{1} \arrow[dr, tail, "u"] \arrow[drr, tail, "p"] \arrow[ddr, "id_{\mathbf{1}}"'] & & \\
& U \arrow[r, tail, "f"'] \arrow[d, dashed, "!"] & \Omega^{\mathbf{O}} \arrow[d, "\chi_{U,f}"] \\
& \mathbf{1} \arrow[r, tail, "\mathrm{t}"] & \Omega
\end{tikzcd}
$$

In the same topos (exponentiation, since cartesian closed):

- The object $\Omega^{\mathbf{O}}$ corresponds to the collection of all properties.
- The centration of f as $\chi_{U,f}$ can also be interpreted as an object.
- The connection between $\mathbf{O}$ and $(\mathbf{O} \to \Omega) \to \Omega$?

The Evaluation Arrow *ev*

$$\begin{tikzcd}
\mathbf{1} \arrow[d, "{p,o}"] \arrow[dr] & \\
\Omega^{\mathbf{O}} \times \mathbf{O} \arrow[r, "ev"] & \Omega \\
C \times \mathbf{O} \arrow[u, "{(\hat{g}, id_{\mathbf{O}})}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

In the same topos (product, since cartesian closed):

- The composition $ev \circ \langle p, o \rangle$ correspond to a truth-value $1 \to \Omega$.

References

The End